SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2009

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading Time – 5 Minutes
- Working time – 120 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Start each **NEW** question in a separate answer booklet.
- Marks may **NOT** be awarded for messy or badly arranged work.
- All answers must be given in exact simplified form unless otherwise stated.
- All necessary working should be shown in every question.

Total Marks – 84

- Attempt questions 1-7.

Examiner: *D.McQuillan*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

START EACH NEW QUESTION ON A NEW ANSWER BOOKLET

Question 1 (12 marks) **Marks**

(a) Solve $x(3-2x)>0$. **2**

(b) Find $\frac{d}{dx}\left(e^{-x}\cos^{-1}x\right)$ **2**

(c) The remainder when x^3+ax^2-3x+5 is divided by $(x+2)$ is 11. Find the value of a. **2**

(d) Find the general solution of $2\cos x+\sqrt{3}=0$. **2**

(e) Solve $\frac{x^2-9}{x}\geq 0$. **2**

(f) Find $\int_0^2\left(4+x^2\right)^{-1}dx$. **2**

End of Question 1

START EACH NEW QUESTION ON A NEW ANSWER BOOKLET

Question 2 (12 marks) **Marks**

(a) Use the substitution $x = \ln u$ to find $\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$. **3**

(b) Use one application of Newton's method to find an approximation to the root of the equation $\cos x = x$ near $x = 0.5$. Give you answer correct to two decimal places. **3**

(c) The curves $y = e^{2x}$ and $y = 1 + 4x - x^2$ intersect at the point (0, 1). Find the angle between the two curves at this point of intersection. **3**

(d) **3**

(i) In how many ways can a committee of 2 Englishmen, 2 Frenchmen and 1 American be chosen from 6 Englishmen, 7 Frenchmen and 3 Americans.

(ii) In how many of these ways do a particular Englishman and a particular Frenchman belong to the committee?

End of Question 2

START EACH NEW QUESTION ON A NEW ANSWER BOOKLET

Question 3 (12 marks) **Marks**

(a) Evaluate $\cos\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$. **1**

(b) **3**

(i) Expand $\cos(\alpha+\beta)$.

(ii) Show that $\cos 2\alpha = 1 - 2\sin^2\alpha$.

(iii) Evaluate $\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{x^2}$.

(c) If $\alpha = \tan^{-1}\left(\frac{5}{12}\right)$ and $\beta = \cos^{-1}\left(\frac{4}{5}\right)$, calculate the exact value of $\tan(\alpha-\beta)$. **2**

(d) A and B are points (–1, 7) and (5, –2); P divides AB internally in the ratio $k:1$. **3**

(i) Write down the coordinates of P in terms of k.

(ii) If P lies on the line $5x-4y=1$, find the ratio of AP:PB.

(e) Use mathematical induction to prove that **3**

$$1\times 1!+2\times 2!+3\times 3!+\cdots+n\times n!=(n+1)!-1,$$

where n is a positive integer.

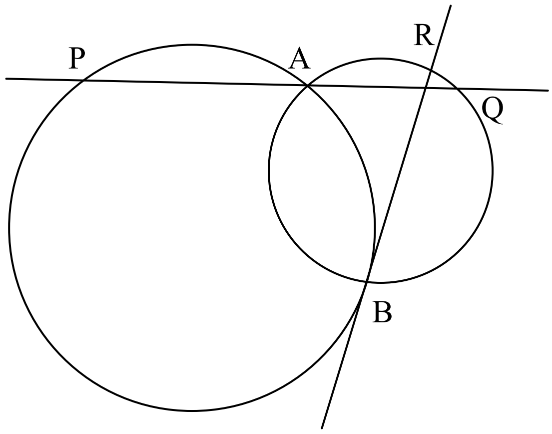
End of Question 3

START EACH NEW QUESTION ON A NEW ANSWER BOOKLET

Question 4 (12 marks) **Marks**

(a) If $\dfrac{dy}{dx} = 1 + y$ and when $x = 0$, $y = 2$ find y as a function of x. **3**

(b) Two circles cut at A and B. A line through A meets one circle at P and the other at Q. BR is a tangent to circle ABP and R lies on circle ABQ. Prove that PB||QR. **3**



(c) The area bounded by the curve $y = \sin^{-1} x$ the y axis and $y = \dfrac{\pi}{2}$ is rotated about the y axis. Find the volume of the solid generated. **3**

(d) A particle moves in a straight line from a position of rest at a fixed origin O. Its velocity is v when displacement from O is x. If its acceleration is $\dfrac{1}{(x+3)^2}$, find v in terms of x. **3**

End of Question 4

START EACH NEW QUESTION ON A NEW ANSWER BOOKLET

Question 5 (12 marks) **Marks**

(a) The speed v ms^{-1} of a particle moving along the x axis is given by $v^2 = 24 - 6x - 3x^2$, where x m is the distance of the particle from the origin. **4**

(i) Show that the particle is executing Simple Harmonic Motion.

(ii) Find the amplitude and the period of motion.

(b) Five Jovians and four Martians are sitting around discussing galactic peace. **5**

(i) In how many ways can they be arranged around the table?

(ii) If Marvin the Martian will not sit next to any of the Jovians, how many arrangements are possible?

(iii) If all the Jovians sit together and all the Martians sit together and Marvin will still not sit next to a Jovian, how many arrangements are possible?

(c) If one root of $x^3 + px^2 + qx + r = 0$ equals the sum of the two other roots, prove that $p^3 + 8r = 4pq$. **3**

End of Question 5

START EACH NEW QUESTION ON A NEW ANSWER BOOKLET

Question 6 (12 marks) **Marks**

(a) $f(x) = \cos x - \sqrt{3}\sin x$, where $0 \le x \le 2\pi$. **3**

(i) Write $f(x)$ is the form $R\cos(x+\alpha)$ where $R > 0$ and α is in the first quadrant.

(ii) Hence solve $f(x) = 1$.

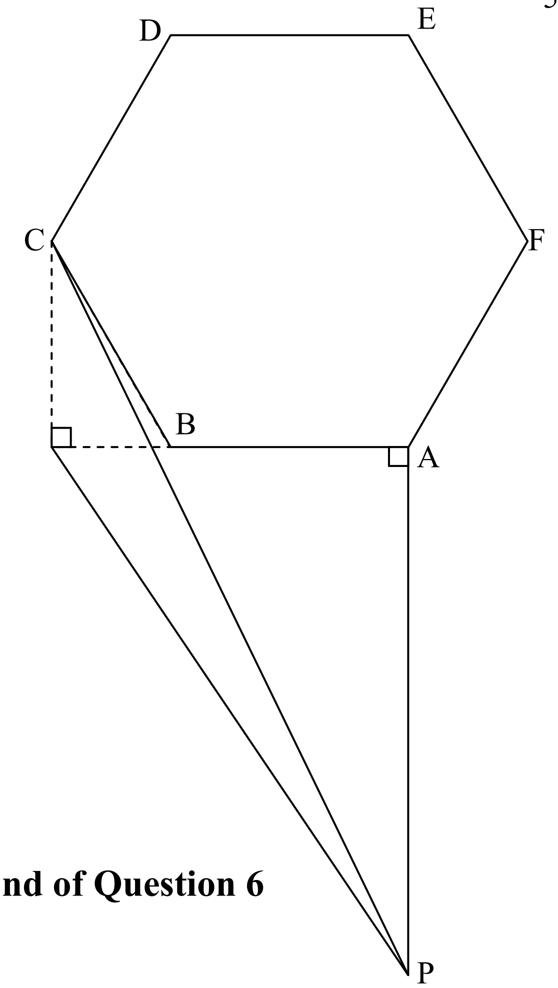
(b) Wheat falls from an auger onto a conical pile at the rate of 20 cm^{3}s^{-1}. The radius of the base of the pile is always equals to half its height. **5**

(i) Show that $V = \dfrac{1}{12}\pi h^3$ and hence find $\dfrac{dh}{dt}$.

(ii) Find the rate at which the pile is rising when it is 8 cm deep, in terms of π.

(iii) Find the time taken for the pile to reach a height of 8 cm.

(c) In a horizontal triangle APB, AP = 2AB, and the angle A is a right angle. On AB stands a vertical and regular hexagon ABCDEF. Prove that PC is inclined to the horizontal at an angle whose tangent is $\dfrac{\sqrt{3}}{5}$. **4**



End of Question 6

START EACH NEW QUESTION ON A NEW ANSWER BOOKLET

Question 7 (12 marks) **Marks**

(a) Use mathematical induction to prove that $\cos(\pi n) = (-1)^n$, where n is a positive integer. **2**

(b) **3**

(i) Find the largest possible domain of positive values for which $f(x) = x^2 - 5x + 13$ has an inverse.

(ii) Find the equation of the inverse function, $f^{-1}(x)$.

(c) The straight line $y = mx + b$ meets the parabola $x^2 = 4ay$ at the points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$.

7

(i) Find the equation of the chord PQ and hence or otherwise show that $pq = -\dfrac{b}{a}$.

(ii) Prove that $p^2 + q^2 = 4m^2 + \dfrac{2b}{a}$.

(iii) Given that the equation of the normal to the parabola at P is $x + py = 2ap + ap^3$ and that N, the point of intersection of the normals at P and Q, has coordinates

$$\left[-apq(p+q), a\left(2 + p^2 + pq + q^2\right)\right],$$

express these coordinates in terms of a, m and b.

(iv) Suppose that the chord PQ is free to move while maintaining a fixed gradient. Find the locus of N and show that this locus is a straight line.

Verify that this line is a normal to the parabola.

End of Question 7

End of Exam

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

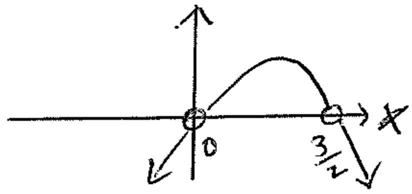
$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

Question 1.

(a) $x(3-2x)>0$



$\therefore \boxed{0<x<\frac{3}{2}}$

(b)

$$\frac{d}{dx}\left[e^{-x}.\cos^{-1}x\right]$$

$$= e^{-x}.\frac{-1}{\sqrt{1-x^2}}+\cos^{-1}x.-e^{-x}$$

$$= \frac{-e^{-x}}{\sqrt{1-x^2}} - e^{-x}\cos^{-1}x$$

(c) let $P(x)=x^3+ax^2-3x+5$

then $P(-2)=11$ (Rem. Th)

$\Rightarrow (-8)+4a+6+5=11$

$4a=8$

$\boxed{a=2}$

(d) $2\cos x+\sqrt{3}=0$

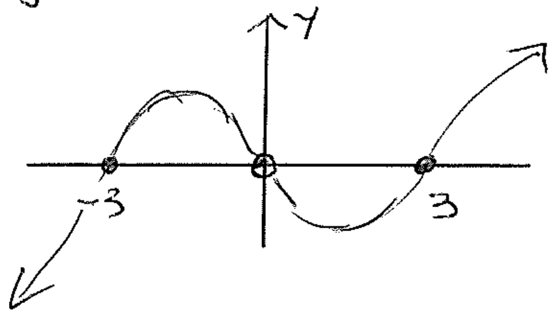
$\Rightarrow \cos x=-\frac{\sqrt{3}}{2}$

i.e. $\cos x=\cos\frac{5\pi}{6}$

$\therefore \boxed{x=2n\pi\pm\frac{5\pi}{6}}$

(e) $\frac{x^2-9}{x}\geq 0 \quad [x\neq 0]$

$\times$ by $x^2 \Rightarrow x(x^2-9)\geq 0$



$\boxed{-3\leq x<0 \quad \text{or} \quad x\geq 3}$

(f) $\int_0^2 \frac{dx}{4+x^2}$

$$=\frac{1}{2}\left[\tan^{-1}\frac{x}{2}\right]_0^2$$

$$=\frac{1}{2}\left[\tan^{-1}1-\tan^{-1}0\right]$$

$$=\frac{1}{2}\left[\frac{\pi}{4}-0\right]$$

$$=\frac{\pi}{8}$$

QUESTION 2

(a) $x = \ln u$ $\quad \frac{du}{dx} = e^x$

$u = e^x$

$dx = \frac{du}{e^x}$

$= \frac{du}{u}$

$$\int \frac{\cancel{u}}{\sqrt{1-u^2}} \frac{du}{\cancel{u}}$$

$= \sin^{-1} u + c$

$= \sin^{-1} e^x + c$

(b) $\cos x - x = 0$

$f(x_1) = \cos 0.5 - 0.5 = 0.378$

$f'(x_1) = -\sin 0.5 - 1 = -1.479$

$x_2 = 0.5 - \frac{0.378}{-1.479}$

$= 0.7556$

$= 0.76$ 2d.p.

(c) $m_1 = 2e^{2x}$ $m_2 = 4 - 2x$

$x=0$ $m_1 = 2$, $m_2 = 4$

$\tan\theta = \left|\frac{2-4}{1+2.4}\right|$

$= 2/9$

$\theta = 12^\circ 32'$

(d) (i) ${}^6C_2 \times {}^7C_2 \times 3 = 945$

(ii) ${}^5C_1 \times {}^6C_1 \times 3$

$= 90$

3 unit Trial ASC 2009

3 (a) $\cos\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$

$= \cos\left(-\frac{\pi}{6}\right)$

$= \cos\frac{\pi}{6}$ even fn

$= \frac{\sqrt{3}}{2}$. (1)

(b) (i) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ (1)

(ii) let $\beta = \alpha$. $\cos(\alpha+\alpha) = \cos\alpha\cos\alpha - \sin\alpha\sin\alpha$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

using $\sin^2\alpha + \cos^2\alpha = 1$.

$\cos 2\alpha = 1 - \sin^2\alpha - \sin^2\alpha$

$= 1 - 2\sin^2\alpha$. (1)

(iii) $\lim_{x\to 0}\frac{1-\cos 2x}{x^2} = \lim_{x\to 0}\frac{2\sin^2 x}{x^2} = 2\lim_{x\to 0}\frac{\sin x}{x}\times\frac{s}{x}$

since $\cos 2x = 1 - 2\sin^2 x$

$2\sin^2 x = 1 - \cos 2x$

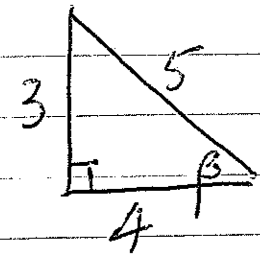
$= 2\times 1\times 1$

$= 2$. (1)

(c) $\alpha = \tan^{-1}\left(\frac{5}{12}\right)$ $\beta = \cos^{-1}\left(\frac{4}{5}\right)$

$\tan\alpha = \frac{5}{12}$ $\cos\beta = \frac{4}{5}$

So $\tan\beta = \frac{3}{4}$.



So $\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} = \frac{\frac{5}{12}-\frac{3}{4}}{1+\frac{5}{12}\times\frac{3}{4}} = \frac{-\frac{1}{3}}{1\frac{5}{16}}$ (2)

$= \frac{-16}{63}$

check $(-1,7)$ $(5,-2)$ $\underset{m}{17}:\underset{n}{16}$.

$$\frac{17\times5+16\times-1}{17+16}, \frac{17\times-2+16\times7}{17+16}$$

$$= \frac{69}{33} = 2\frac{1}{11} \checkmark \quad \frac{78}{33} = 2\frac{4}{11}. \checkmark$$

(e) $1\times1! + 2\times2! + 3\times3! + \cdots + n\times n! = (n+1)! - 1$
for n a positive integer

step 1 let $n=1$, $LHS = 1\times1! = 1$
$RHS = (1+1)! - 1 = 2! - 1 = 2-1 = 1$

So $n=1$ is true.

step 2 Assuming it is true for $n=k$,
$1\times1! + 2\times2! + 3\times3! + \cdots + k\times k! = (k+1)! - 1$
we must prove that for $n=k+1$,
$1\times1! + 2\times2! + 3\times3! + \cdots + k\times k! + (k+1)\times(k+1)!$
$= (k+2)! - 1$.

LHS $(k+1)! - 1 + (k+1)(k+1)!$

$= (k+1)! \left[1 + k + 1\right] - 1$

$= (k+1)! \left[k+2\right] - 1$

$= (k+2)! - 1.$

$=$ RHS.

<u>step 3</u> Hence the statement is true for $n = k+1$
By the principle of math induction it is true for all $n \geq 1$.

3

4) a) $\frac{dy}{dx} = 1 + y$

$\frac{dx}{dy} = \frac{1}{1+y}$

$x = \ln(1+y) + C$

when $x=0$, $y=2$

$0 = \ln(3) + C$

$C = -\ln 3$

$x = \ln(1+y) - \ln 3$

$x = \ln\left(\frac{1+y}{3}\right)$

$\frac{1+y}{3} = e^x$

$1+y = 3e^x$

$y = 3e^x - 1$

b) let $R\hat{Q}A = x$

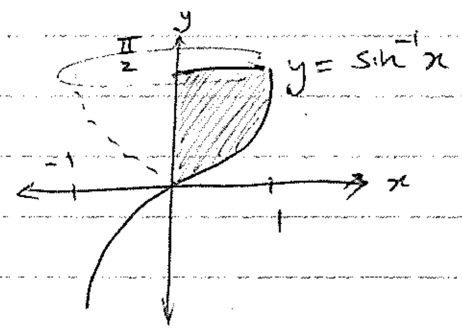
$A\hat{B}R = x$ (angles in same segment)

$B\hat{P}A = x$ (alternate segment theorem)

since alternate angles equal ($R\hat{Q}P = Q\hat{P}B$)

$PB \parallel QR$

c)



$x = \sin y.$

$$V = \pi \int_a^b x^2 \, dy$$

$$V = \pi \int_0^{\frac{\pi}{2}} \sin^2 y \, dy$$

$$V = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} (1 - \cos 2y) \, dy$$

$$V = \frac{\pi}{2} \left[y - \frac{1}{2} \sin 2y \right]_0^{\frac{\pi}{2}}$$

$$V = \frac{\pi}{2} \left[\frac{\pi}{2} - \cancel{\frac{1}{2} \sin 2\left(\frac{\pi}{2}\right)}^{0} - \left(0 - \cancel{\frac{1}{2} \sin 2(0)}^{0}\right)\right]$$

$$V = \frac{\pi^2}{4} \text{ units}^3.$$

d) $$a = \frac{1}{(x+3)^2}$$

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = (x+3)^{-2}$$

$$\frac{1}{2}v^2 = \frac{(x+3)^{-1}}{-1 \times 1} + C$$

$$\frac{1}{2}v^2 = -\frac{1}{x+3} + C$$

when $x = 0, v = 0$

$$0 = -\frac{1}{3} + C$$

$$C = \frac{1}{3}$$

$$\frac{1}{2}v^2 = \frac{1}{3} - \frac{1}{x+3}$$

$$v^2 = 2\left(\frac{1}{3} - \frac{1}{x+3}\right)$$

$$v = \pm\sqrt{2\left(\frac{1}{3} - \frac{1}{x+3}\right)}$$

but acceleration is always positive. & since it starts from rest.

$$V = \sqrt{2\left(\frac{1}{3} - \frac{1}{x+3}\right)} \quad \underline{\text{OR}} \quad \sqrt{\frac{2x}{3(x+3)}}$$

Question 5

(a) (i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \ddot{x} = -3-3x$

$= -3(x+1)$

Let $X = x+1$, so $\ddot{X} = \ddot{x}$

$\therefore \quad \ddot{X} = -3\cancel{x}$ [2]

Hence, Simple Harmonic Motion.

(ii) From above, $n^2 = 3$

$V^2 = 3(8-2x-x^2)$

$= 3(8-(x^2+2x+1)+1)$

$= 3(9-(x+1)^2)$

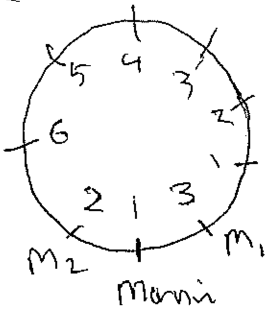
$\therefore v^2 = 3(9-X^2)$

$\therefore a^2 = 9 \qquad T = \frac{2\pi}{\sqrt{3}}$ [2]

$a = 3$

(b) (i) $8! = 40320$ ways [1]

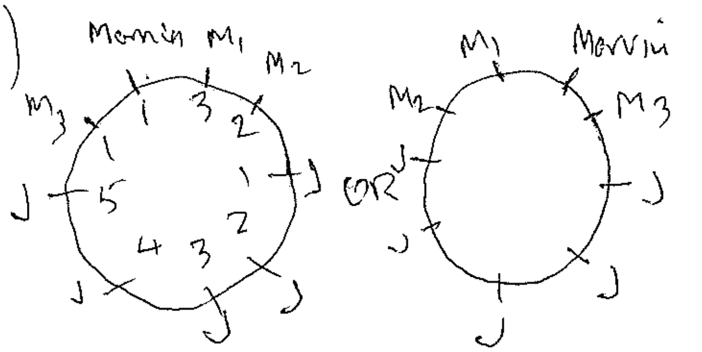
(ii)



$2 \times 3 \times 6! = 4320$ ways

[2]

(iii)



$\therefore 2 \times (3 \times 2 \times 5!) = 1440$ ways

[2]

(c) $x^3 + px^2 + qx + r = 0$

Let roots be $\alpha, \beta, \alpha+\beta$

Now $-p = 2(\alpha+\beta)$

$q = \alpha\beta + (\alpha^2 + \alpha\beta) + (\alpha\beta + \beta^2)$

$= 3\alpha\beta + \alpha^2 + \beta^2$

$-r = \alpha\beta(\alpha+\beta)$

$= \alpha^2\beta + \alpha\beta^2$ [1]

RTP: $p^3 + 8r = 4pq$

LHS $= -8(\alpha+\beta)^3 + (-8(\alpha^2\beta + \alpha\beta^2))$

$= -(8(\alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3) + 8(\alpha^2\beta + \alpha\beta^2))$

$= -(8\alpha^3 + 32\alpha^2\beta + 32\alpha\beta^2 + 8\beta^3)$

RHS $= -8(\alpha+\beta)(3\alpha\beta + \alpha^2 + \beta^2)$

$= -8(3\alpha^2\beta + \alpha^3 + \alpha\beta^2 + 3\alpha\beta^2 + \alpha^2\beta + \beta^3)$

$= -(8\alpha^3 + 32\alpha^2\beta + 32\alpha\beta^2 + 8\beta^3)$

$=$ LHS as required.

[2]

Alternatively

$\alpha + \beta = -\frac{p}{2}$

But $\alpha+\beta$ is a root.

$\therefore P\left(-\frac{p}{2}\right) = 0$

$\left(\frac{-p}{2}\right)^3 + p\left(-\frac{p}{2}\right)^2 + q\left(\frac{-p}{2}\right) + r = 0$

$-\frac{p^3}{8} + \frac{p^3}{4} + \left(\frac{-pq}{2}\right) + r = 0$

$\frac{p^3}{8} + \left(\frac{-pq}{2}\right) + r = 0$

$\therefore p^3 + 8r = 4pq$ [3]

Question (6)

(a)

(i) $\cos x - \sqrt{3}\sin x = R\cos(\alpha + x)$

$R = \sqrt{1+3}$ =	2
$\tan\alpha = \frac{\sqrt{3}}{1} \Rightarrow$	$\alpha = \frac{\pi}{3}$.

1

(ii) $2\cos\left(x + \frac{\pi}{3}\right) = 1$

$\therefore \cos\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$

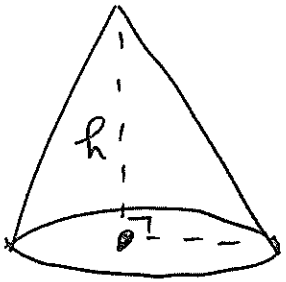
$x + \frac{\pi}{3} = \frac{\pi}{3},\ \frac{5\pi}{3},\ \frac{7\pi}{3}$

2 $\therefore x = 0,\ \frac{4\pi}{3},\ 2\pi$

3

(b)

(i)



$r = \frac{h}{2}$

$V = \frac{\pi}{3} r^2 h = \frac{\pi}{3} \cdot \frac{h^2}{4} \cdot h$

$\therefore V = \frac{\pi h^3}{12}$ 1

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$= \frac{1}{(dV/dh)} \times \frac{dV}{dt}$

$= \frac{4}{\pi h^2} \times 20$ 2

$= \frac{80}{\pi h^2}$ cm/s.

(ii) $\frac{\cancel{80}\ 10}{\pi \times \cancel{64}\ 8} = \frac{5}{4\pi}$ cm/s.

(0.398) 1

(iii) $\frac{dh}{dt} = \frac{80}{\pi h^2}$

$\therefore \frac{dt}{dh} = \left(\frac{\pi}{80}\right) h^2$

$t = \frac{\pi}{80}\int_0^8 h^2\, dh$ 1

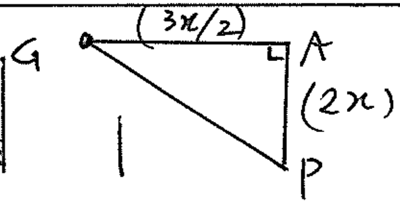
$= \left[\frac{\pi h^3}{240}\right]_0^8$

$= \frac{32\pi}{15}$ sec

$\doteqdot 6.7\ \text{sec}^2.$

5

$\boxed{AG = \frac{x}{2} + x = 3x/2}$



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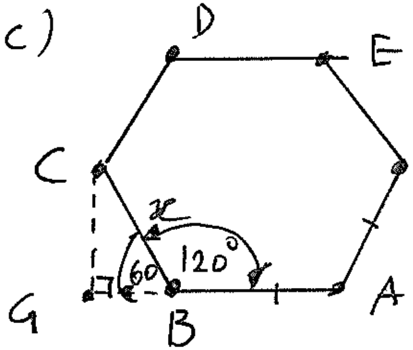
(c)



4

$AB = x = CB$

Express $\begin{cases} BG \\ CG \end{cases}$ in terms of x

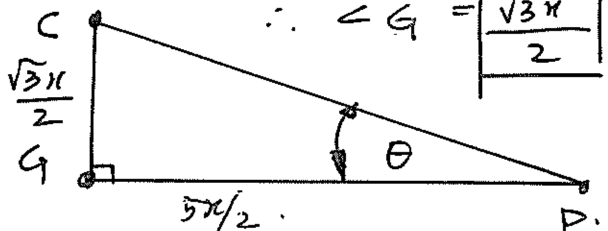
$\angle CBA = 120°$ (regular hex)

$\Longrightarrow CBG = 60°$

$\frac{BG}{x} = \cos 60° = \boxed{\frac{x}{2}}$ 1

$\frac{CG}{x} = \frac{\sqrt{3}}{2}$

$\therefore CG = \boxed{\frac{\sqrt{3}x}{2}}$



$GP = \sqrt{4x^2 + \frac{9x^2}{4}}$

$= \sqrt{\frac{25}{4}x^2} = \frac{5x}{2}.$ 2

$\therefore \tan\theta = \frac{\sqrt{3}\cancel{x}}{\cancel{2}} \times \frac{\cancel{2}}{5\cancel{x}}$

$= \sqrt{3}/5$

2009 Mathematics Extension 1 Trial HSC: **Question 7** solutions

7. (a) Use mathematical induction to prove that $\cos(\pi n) = (-1)^n$, where n is a positive integer. $\boxed{2}$

Solution: Test for $n = 1$:

$$\begin{aligned} \text{L.H.S.} &= \cos\pi, & \text{R.H.S.} &= (-1)^1, \\ &= -1. & &= -1. \end{aligned}$$

$\therefore$ True when $n = 1$.

Now assume true when $n = k$, some particular integer,

i.e. $\cos(\pi k) = (-1)^k$.

Then test for $n = k + 1$, *i.e.* $\cos\big(\pi(k+1)\big) = (-1)^{k+1}$.

$$\begin{aligned} \text{L.H.S.} &= \cos\big(\pi(k+1)\big), \\ &= \cos(\pi k + \pi), \\ &= \cos\pi k \cos\pi - \sin\pi k \sin\pi, \\ &= (-1)^k.(-1) - 0, \text{ using the assumption,} \\ &= (-1)^{k+1}, \\ &= \text{R.H.S.} \end{aligned}$$

$\therefore$ True for all $n \geqslant 1$ by the principle of mathematical induction.

(b) (i) Find the largest possible domain of positive values for which $f(x) = x^2 - 5x + 13$ has an inverse. $\boxed{3}$

Solution:
$$\begin{aligned} f'(x) &= 2x - 5, \\ 2x - 5 &= 0 \text{ when } x = 5/2. \end{aligned}$$

$\therefore$ Function is one-one if $x > 5/2$.

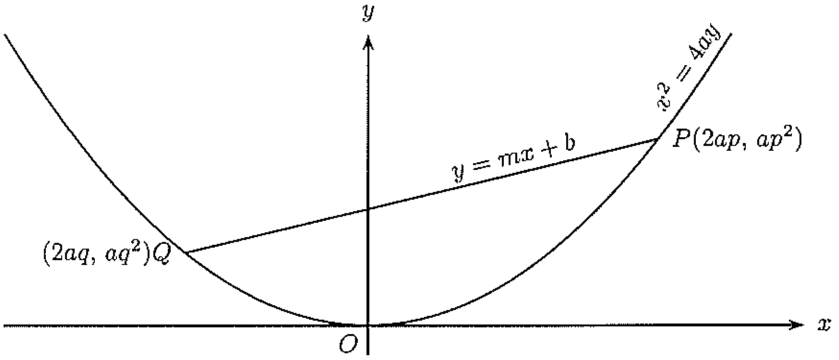
(ii) Find the equation of the inverse function, $f^{-1}(x)$.

Solution: Put
$$\begin{aligned} x &= y^2 - 5y + 13, \\ &= y^2 - 5y + \tfrac{25}{4} + 13 - \tfrac{25}{4}, \\ x - \tfrac{27}{4} &= (y - 5/2)^2, \\ y - 5/2 &= \frac{\pm\sqrt{4x - 27}}{2}, \\ y &= \frac{5 \pm \sqrt{4x - 27}}{2}, \\ \textit{i.e. } f^{-1}(x) &= \frac{5 + \sqrt{4x - 27}}{2}, \text{ taking the positive root as } f^{-1}(x) > 5/2. \end{aligned}$$

(c) The straight line $y = mx + b$ meets the parabola $x^2 = 4ay$ at the points $P(2ap,\ ap^2)$ and $Q(2aq,\ aq^2)$. $\boxed{7}$

(i) Find the equation of the chord PQ and hence or otherwise show that $pq = -\frac{b}{a}$.

Solution:



$$\begin{aligned}
y - ap^2 &= \frac{ap^2 - aq^2}{2ap - 2aq}(x - 2ap), \\
&= \frac{p+q}{2}(x - 2ap), \\
2y - 2ap^2 &= (p+q)x - 2ap^2 - 2apq, \\
2y &= (p+q)x - 2apq \text{ is the equation of } PQ.
\end{aligned}$$

This is the same line as $y = mx + b$ so $b = -apq$ and thus $pq = -\dfrac{b}{a}$.

(ii) Prove that $p^2 + q^2 = 4m^2 + \dfrac{2b}{a}$.

Solution: $m = \dfrac{p+q}{2}$,

$$\begin{aligned}
\therefore \text{ R.H.S.} &= 4\left(\frac{p+q}{2}\right)^2 + 2(-pq), \\
&= p^2 + 2pq + q^2 - 2pq, \\
&= p^2 + q^2, \\
&= \text{L.H.S.}
\end{aligned}$$

(iii) Given that the equation of the normal to the parabola at P is $x + py = 2ap + ap^3$ and that N, the point of intersection of the normals at P and Q, has coördinates

$$\left[-apq(p+q),\ a(2 + p^2 + pq + q^2)\right],$$

express these coördinates in terms of a, m and b.

Solution: Now $-apq = b,\ p + q = 2m,\ p^2 + q^2 = 4m^2 + 2b/a$.

$$\begin{aligned}
\therefore\ x_N = 2bm, \quad y_N &= a(2 + 4m^2 + 2b/a - b/a), \\
&= a(2 + 4m^2 + b/a).
\end{aligned}$$

$\therefore\ N : [2bm,\ 2a + 4am^2 + b]$

(iv) Suppose that the chord PQ is free to move while maintaining a fixed gradient. Find the locus of N and show that this locus is a straight line.

Verify that this line is a normal to the parabola.

Solution: Method 1—

$b = \dfrac{x}{2m}$,

$y = \dfrac{x}{2m} + 2a + 4am^2$ which is the locus of N

and a straight line with a slope of $1/2m$.

Rewriting, $x - 2my = -4am - 8am^3$,

then let $p = -2m$ so that $x + py = 2ap + ap^3$

which is in the form of a normal to the parabola $x^2 = 4ay$.

Solution: Method 2—

$b = \dfrac{x}{2m}$,

$y = \dfrac{x}{2m} + 2a + 4am^2$ which is the locus of N

and a straight line with a slope of $\frac{1}{2m}$.

Where this locus of N meets the parabola $x^2 = 4ay$,

$$\begin{aligned} x^2 &= 4a\left(\frac{x}{2m} + 2a + 4am^2\right), \\ mx^2 - 2ax - 8a^2m + 16a^2m^3 &= 0. \\ x &= \frac{2a \pm \sqrt{4a^2 + 4a^2\left(8m^2 + 16m^4\right)}}{m}, \\ &= \frac{a \pm a\sqrt{1 + 8m^2 + 16m^4}}{m}, \\ &= \tfrac{a}{m}\left(1 \pm (1 + 4m^2)\right), \\ &= \tfrac{a}{m}\left(2 + 4m^2\right) \text{ or } \tfrac{a}{m}\left(-4m^2\right). \end{aligned}$$

In the limiting case when $x = -4am$, $p = q$ and $-4am = 2ap$,

$$\therefore\ p = -2m.$$

So the slope of the normal at this point is $\frac{1}{2m}$ which is the slope of the locus of N.